⇒ No lecture on Monday (3/7)!!
(CEC Open House)

* Goal is to have midterm on
  Week 3/30
  
  (after Proj #3 is submitted)
Plane Waves

\[ p(x,y,z,t) \Rightarrow p(z,t) \]

\[ C \]

\[ z \text{ is the propagation axis} \]

\[ \text{normalized to the face of the transducer} \]

Wave Equation:

\[ \nabla^2 p = \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} = \frac{\delta p}{\delta t^2} \]

2nd derivative w/ respect to space (with a constant) 2nd deriv. w/ respect to time
\[ p(z_i,t) = \phi_f(t - \frac{z_i}{c}) + \phi_0(t - \frac{z_i}{c}) \]

\[ \phi_z \quad \begin{array}{cc} \phi_f & \phi_0 \\ \phi_f & \phi_f \end{array} \]

\[ \Rightarrow \text{both terms must be twice differentiable} \]
\[ \Rightarrow \text{each term must satisfy the wave efc} \]

\[ \Rightarrow \text{we will discard } \phi_0 \]

\[ \Rightarrow p(z_i,t) = \phi_f(z_i,t) \]

\[ \Rightarrow \text{Assume sinusoidal} \]
\[ p(z,t) = \cos \left( k(z - ct) \right) \]

- Spatial term

\[ = \cos \left( kct - kz \right) \]

\[ \omega \ldots \text{oscillation in time} \]

\[ = \cos \left( \omega t - kz \right) \]

- For a fixed \( t \), the sinusoid will vary in time with \( \omega = 2\pi f \)

\[ \left( f = \frac{k c}{2\pi} \right) \]

- For a fixed \( t \), the sinusoid will vary in space with a spatial frequency \( k \) [wave number]

\[ \left( k = \frac{2\pi}{\lambda}, \lambda = \frac{c}{f} \right) \]
For ultrasound (and other reflection techniques) is to create a pulse that will code "close" to a sinusoid.

\[ \lambda \]

- Present several cycles of the sinusoid to generate \( f, \lambda \)
- Typically windowed to "smooth" the onset/offset

Consider \( f = 359 \text{MHz}, \ c = 1540 \text{m/s} \)

\( \lambda = \frac{c}{f} \approx 44 \text{mm} \) 

Typically, pulse width (duration) will be several cycles 

\[ 5 \text{ or } 10 \text{ common} \]
Spherical Waves

- In an isotropic medium... any small disturbance = point source.
  - will subsequently produce a propagating wave (uniformly propagating)
  - this wave only depends on
    \[ t \neq r = \sqrt{(ax)^2 + (by)^2 + (cz)^2} \]

\[ \Rightarrow \text{Spherical version of the wave equation} \]
\[ \frac{1}{r} \frac{\partial^2 (rP)}{\partial r^2} = \frac{1}{c^2} \frac{\partial^2 P}{\partial t^2} \]
\[ p(r, t) = \frac{1}{r} \phi_0 (t - \frac{r}{c}) + \frac{1}{r} \phi_0 (t - \frac{r}{c}) \]

**outbound**

\[ \text{surface area}
\]

\[ \text{increases with } r, \]

\[ \text{so pressure drops} \]

\[ \Rightarrow \text{At our transducer, the incident return wave is spherical with } r = \frac{c}{2} \]

\[ \text{like a plane wave but with } \frac{1}{r} \text{ dependence} \]
Wave Energy

Particles in motion have kinetic energy:

Per unit volume, the density of kinetic energy:

\[ W_k = \frac{1}{2} \rho V^2 \]

- \( \rho \): propagation velocity
- \( \rho \): density of the medium

Potential energy:

\[ W_p = \frac{1}{2} KP^2 \]

- \( K \): pressure
- \( P \): compressibility of medium

Total Energy Density:

\[ W = W_k + W_p \]
Energy with the wave:

\[
\text{Intensity (per unit area)} \rightarrow I = p \nu
\]

Pressure

\[
I = \frac{p^2}{z}
\]

Impedance to wave propagation in the medium

Due to conservation of energy:

\[
\frac{\partial I}{\partial z} + \frac{\partial W}{\partial t} = 0
\]

After we inject the energy into the form, energy cannot change.
What happens at a tissue boundary?

Critical Angle $\Theta_C = \sin^{-1} \left( \frac{c_1}{c_2} \right)$

Snell's Law:

$$\frac{\sin \Theta_i}{\sin \Theta_t} = \frac{c_1}{c_2}$$

If $\frac{c_2}{c_1} \sin \Theta_i = \sin \Theta_t > 1 \Rightarrow \text{All energy is reflected}$
Transmission & Reflection Coefficients

Boundary condition: Particle velocities must result in a net balance
\[ \Rightarrow \text{boundary cannot shift} \]

\[ \text{medium}^*1 \rightarrow \text{medium}^*2 \]
\[ v_r \quad \text{normal terms must equal} \]
\[ |V_i| \cos \theta_i = |V_r| \cos \theta_r + |V_t| \cos \theta_t \]
\[ p = \text{constant} \]

\[ \Rightarrow \text{pressure cannot be discontinuous} \ldots \ p_t - p_r = p_i \]
\[
\frac{P_i \cos \Theta_i}{z_1} = \frac{P_r \cos \Theta_r}{z_1} + \frac{P_t \cos \Theta_t}{z_2}
\]

\[
R = \frac{P_r}{P_i} = \frac{z_2 \cos \Theta_i - z_1 \cos \Theta_t}{z_2 \cos \Theta_i + z_1 \cos \Theta_t}
\]

\[
T = \frac{P_t}{P_i} = \frac{2 z_2 \cos \Theta_i}{z_2 \cos \Theta_i + z_1 \cos \Theta_t}
\]

\[
T - R = 1 \quad (I = \frac{P^2}{z})
\]

PREFER to have these in terms of ENERGY.
\[ R_I = \frac{I_r}{I_i} = \left( \frac{z_2 \cos \theta_i - z_1 \cos \theta_t}{z_2 \cos \theta_i + z_1 \cos \theta_t} \right)^2 \]

\[ T_I = \frac{I_t}{I_i} = \frac{4z_1 z_2 \cos^2 \theta_i}{[z_2 \cos \theta_i + z_1 \cos \theta_t]^2} \]

In ultrasound, we assume normal incidence:

\[ \Rightarrow \theta_i = \theta_r = \theta_t = 0^\circ \quad (\text{all } \cos \rightarrow 1) \]

\[ R_I = \left( \frac{z_2 - z_1}{z_2 + z_1} \right)^2 \quad T_I = \frac{4z_1 z_2}{(z_2 + z_1)^2} \]