Convolution Back Projection

\[ f(x, y) = \lim_{\omega \to 0} \int_{-\infty}^{\infty} |p| \cdot p^c(p) e^{j\omega p} \, dp \]

\[ g_0(t) = \int_{-\infty}^{\infty} |p| \cdot p^c(p) \, dp \]

\[ g_0(t) = \mathcal{F}^{-1} \{ |p| \cdot p^c(p) \} \]

\[ g_0(t) = h(t) * p^c(t) \]

\[ h(t) \quad \text{response function} \]

\[ p^c(t) \]
\[ f(x, y) = \int_0^\pi g_t(\theta) \, d\theta \quad \text{a line} \]

\[ f(x, y) = \int_0^\pi g_t(x \cos \theta + y \sin \theta) \, d\theta \]

\[ b_\theta(x, y) = g_t(x \cos \theta + y \sin \theta) = g_t(r) \]

\[ f(x, y) = \int_0^\pi b_\theta(x, y) \, d\theta \rightarrow \frac{\pi}{M} \sum_{m=0}^{M-1} \frac{b_{m\pi}(x, y)}{H} \]

"M angles of projections" "weighted projections"

"our projections convolved with h(r)"
\[ f(x, y) \approx \frac{\pi}{M} \sum_{m=0}^{M-1} b_{m,n}(x, y) \]

\[ b_{0}(x, y) \rightarrow g_{0}(r) = h(r) * p_{0}(r) \]

\[ h(r) = \int \psi_{2i-1} \left| \psi \right| \]

\[ H(p) = |p| \]

\[ p_{c} \rightarrow \text{infinite energy, so no } \text{FT!} \]

Any real system must be bandlimited

... truncate the spectrum...
\begin{align*}
\tilde{H}(\rho) &= f_c \text{ rect}\left(\frac{\rho}{2fc}\right) - f_c \Lambda\left(\frac{\rho}{fc}\right) \\
&= f_c \left[ \text{rect}\left(\frac{\rho}{2fc}\right) - \Lambda\left(\frac{\rho}{fc}\right) \right]
\end{align*}

\begin{itemize}
  \item Subtract triangle from rectangle
  \item Both \text{rect}(\cdot) and \Lambda(\cdot) are defined as unit area and centered at \( \rho = 0 \)
\end{itemize}
Both $\text{rect}(\cdot)$ and $\delta(\cdot)$ have known FT:

\[
\text{rect}(\cdot) \Leftrightarrow \text{sinc}(\cdot)
\]

\[
\delta(\cdot) \Leftrightarrow \text{sinc}^2(\cdot)
\]

\[
\hat{h}(\rho) = f_c \left[ \text{rect} \left( \frac{\rho}{2f_c} \right) - \delta \left( \frac{\rho}{f_c} \right) \right]
\]

\[
\hat{h}(r) = \frac{f_c^2}{f_c} \left[ 2\text{sinc}(r2f_c) - \text{sinc}^2(rf_c) \right]
\]

taps An. Convolution with $P_0(\cdot)$ [$P_0(\cdot)$] to obtain $g_0(\cdot)$ & $h_{\text{FFT}}(x,y)$
Summary of CBD:

1) Obtain projections, \( P_0(r) \)

2) Generate and use filter approximation, \( \tilde{W}(r) \)

3) Back-project & sum the filtered projections
"Resolution"

→ How far apart must two "points" be before they can be resolved as distinct?

Typically → "resolution" of an image is often specified as the dimension of the image elements (pixels, voxels).

E.g. 3mm x 3mm x 3mm → voxel size ≤ NOT the actual resolution
Point-Spread Function

*True resolution of an image is a consequence of the shape/size of the PSF

→ Often be given \( FWHM \)

⇒ actual resolving power

ECE 645
STAT 528
Concept: If we have a punctate object (width ≈ 0), how broad does the resulting distribution of "energy" become in the output image?
PSF in CBP

\[ g_0(r) \xrightarrow{\mathcal{F}} \mathcal{F}\{p(r) \ast P_0(r)\} \]

\[ \mathcal{F}\{p(r) \ast P_0(r)\} = \mathcal{F}\{p(r)\} \ast \mathcal{F}\{P_0(r)\} \]

\[ \mathcal{F}\{p(r)\} = \begin{cases} 1 & |f| \leq f_c \\ 0 & |f| > f_c \end{cases} \]

⇒ Detection takes place over the rest of a “window”

⇒ Will produce blur due to lack of high-f information

⇒ Doesn’t exist either...

⇒ Discrete detectors, so quantized
Approx related to $P_a(r)$:

$CT \Rightarrow$
- real detector has non-zero width
  - non-zero height
- detectors are non-ideal
  $\Rightarrow$ not all incident photons produce a response

Shallow angle:
- could pass through before $PE$ occurs
  - no response
  - incorrect location (next detector)
Model this window/area as \( S(p) \)

\[
\hat{g}(x,y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P_0(p) S(p) W(p) \epsilon_{\rho} e^{j2\pi(p\cos\theta + y\sin\theta)} d\rho d\theta
\]

\( \hat{g} \) estimate \( g \)

\( f(x,y) \) ... blurred

Summation elements: \( W(r) \ast P_0(r) \)

\[
\sum \ast p_1 W(p) \ast \hat{g} \{ S(p) P_0(p) \}
\]
$$\hat{P}(r) = \mathcal{F}^{-1} \{ \hat{P}_0(r) S(\rho) W(\rho) \}$$

$$\hat{f}(x,y) = \int_0^{\pi} [\hat{P}(r) \ast h(r)] \, d\theta$$

$$\hat{P}_0(r) = P_0(r) \ast (S(r) \ast W(r))$$

Create $$\tilde{h}(r) = S(r) \ast W(r) \ast \mathcal{F}^{-1} \{ \hat{P}_0(r) \}$$ to sum for

* Blu (PSF) is the inverse Radon transform of $h(r)$!