Project Update

- Have obtained visible human CT data

  => Converting back into Hounsfield units

  => Post-thru project ... Monday?

Given a volume in HU, generate

- 16 CT images ... single point/lines
- 26/36 CT images ... fan-beams, curved detector
- 46/56 CT images ... cone-beam to curved 2D grid
Now general perspective on forward model/recon

Assume:
- a finite object
- density \( f(x,y) \)
- collecting projections
  at every angle, \( \theta \),
  and displacement/translation \( r \)
  \[ \Rightarrow P_\theta(r) \]

At each \( r \), \( P_\theta(r) = \text{projection along/parallel to} \)
\( z \ldots \text{normal to} \ r \)
Sinogram

\[ P_0(r) \text{ plotted as grayscale/intensity, value in a column} \]

\[ (r_0, \phi_0) \]

\[ \text{Sinogram because point produces a sinusoid \ with amplitude } \]
\[ = r_0 \ \text{ & phase } = \phi_0 \ [\text{angle from } +x \text{ axis}] \]
To obtain $P_\theta(r)$:

Define the counter-clockwise rotation matrix $A_\theta$:

$$A_\theta = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

Conversion between orthonormal axes:

$$\begin{bmatrix} x \\ y \end{bmatrix} = A_\theta \begin{bmatrix} r \\ \theta \end{bmatrix}$$

$$\begin{bmatrix} r \\ \theta \end{bmatrix} = A_{-\theta} \begin{bmatrix} x \\ y \end{bmatrix}$$
\[ P_\theta(r) = \int_{-\infty}^{\infty} f(A_\theta[z]) \, dz \]

Radon Transform
Take the Fourier Transform of $P_0(r)$:

$$P_0(\rho) = \int_0^\infty P_0(r) e^{-j2\pi \rho r} \, dr$$

$$= \int_0^\infty \int_{-\infty}^{\infty} f(A_0[z]) \, dz \, e^{-j2\pi \rho r} \, dr$$

Let us switch to $(x, y)$ from $(r, \theta)$:

$$[z] = A_0 \begin{bmatrix} x \\ y \end{bmatrix}$$

Jacobian:

$$dxdy = dzd\theta$$

$$r = x \cos \theta + y \sin \theta$$
\[ P_\theta (\rho) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-j2\pi \rho [x \cos \theta + y \sin \theta]} \, dx \, dy \]

\[ = \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} f(x, y) e^{-j2\pi \rho \cos \theta x} \, dx \right] e^{-j2\pi \rho \sin \theta y} \, dy \]

\[ P_\theta (\rho) = F (\rho \cos \theta, \rho \sin \theta) \]

\( \rho \) polar coordinate FT of \( f(x, y) \)
Fourier Slice Theorem

\[ P_\theta(r) = F(r \cos \theta, r \sin \theta) \]

→ Transformed projection corresponds to a given set of locations in the ZDFT of the object [a line!]

→ locus corresponding to a line through the origin at angle \( \theta \)
Inverse Radon Transform

Fill in as many angles as collected, then do 2DPT⁻¹

Points NOT in a rectilinear grid... NO!!

When points "overlap," tend to overweight middle (low-f)
Typical solution is to apply a 2D kernel to compute/interpolate known values to the rectilinear grid.

\[
\text{Common kernel is:} \quad \frac{\text{Kaiser-Bessel}}{W} = \frac{I_0(\beta \sqrt{1-(\frac{r}{w})^2})}{W}
\]

- \(W\) = width of kernel
- \(r\) = distance from sample

Sum at each rectilinear point!
(normalize to some degree)
Regridding permits use of 2D FFT for image recon...

\[ \Rightarrow \text{Direct Fourier Reconstruction} \]

Typically \underline{not} used... regridding is generally computationally expensive.

\( \text{So mostly due to the shape of the chosen kernel} \)
Consider our object \( \rightarrow 2D \mathbb{T} \):

\[
\hat{f}(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{F}(u, v) e^{j2\pi (ux + vy)} \, du \, dv
\]

Easier in polar coordinates \( \rightarrow \int \int |\rho| d\rho \, d\theta \)

Define

\[
g_\theta(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\rho| \hat{P}_\theta(\rho) e^{j2\pi (x\rho \cos \theta + y\rho \sin \theta)} \, d\rho \, d\theta
\]

\[g_\theta(t) = h(t) * \hat{P}_\theta(t)\]
\[ g_0(t) = \int_{-\infty}^{\infty} |p| P_0(p) e^{i2\pi pt} \, dp \]

\[ g_0(t) = \mathcal{F}^{-1}\{ |p| P_0(p) \} \]

\[ \text{inv. Fourier Transform} \]

\[ P_0(p) \text{ multiplied by a ramp} \]

\[ \text{convolution in time} \]

\[ g_0(t) = \mathcal{F}^{-1}\{ |p| P_0(p) \} \]

\[ \Rightarrow g_0(t) = h(t) \ast P_0(t) \]

\[ \text{approximation of this will facilitate rapid recon...} \]